

Bethe Ansatz for 1D interacting anyons

M.T. Batchelor[†], X.-W. Guan[†] and J.-S. He[‡]

[†]*Department of Theoretical Physics,
Research School of Physical Sciences and Engineering, and
Mathematical Sciences Institute, Australian
National University, Canberra ACT 0200, Australia*

[‡]*Department of Mathematics, University of Science and
Technology of China, Hefei 230026, Anhui, P. R. China*

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Abstract

This article gives a pedagogic derivation of the Bethe Ansatz solution for 1D interacting anyons. This includes a demonstration of the subtle role of the anyonic phases in the Bethe Ansatz arising from the anyonic commutation relations. The thermodynamic Bethe Ansatz equations defining the temperature dependent properties of the model are also derived, from which some groundstate properties are obtained.

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I. INTRODUCTION

A number of 1D models in quantum many-body physics have been solved by the Bethe Ansatz following Bethe's pioneering work on the exact solution of the 1D Heisenberg magnetic spin chain in 1931 [1, 2]. During the past 75 years the Bethe Ansatz has been developed and applied to physical problems such as the 1D δ -function interacting Bose [3, 4] and Fermi [5] gases, the 1D Hubbard model [6] and 2D vertex models [7, 8] (many of these key papers are collected in Ref. [9]). The solution of such models contributed to the development of the celebrated Yang-Baxter equation, which gives the consistency conditions for many-body scattering problems and plays a crucial role in quantum integrable systems [10, 11, 12, 13, 14, 15] and in modern mathematical physics [16, 17]. The related establishment of the quantum inverse scattering method [11, 18, 19] provided widespread applications of the Yang-Baxter equation in low-dimensional quantum systems, such as the Kondo problem [20], the Anderson model [21] and long range interaction systems [22, 23].

A further pioneering application of the Bethe Ansatz was to the 1D δ -function interacting Bose gas [3] which is related to the quantum nonlinear Schrödinger equation. This model provides an important realistic physical description of an interacting 1D Bose gas. It is also arguably one of the most simplest and pedagogic models solved in terms of the Bethe Ansatz. In general the applicability of the Bethe Ansatz depends on the reducibility of the multi-particle scattering matrix to the product of many two-particle scattering matrices. The starting point for the Bethe Ansatz approach in quantum many-body physics is to reduce the eigenvalue problem of the field theoretic hamiltonian into a quantum mechanical many-body problem. The wavefunction of the many-body hamiltonian inherits the statistical signature of the interacting particles, which leads to striking and subtle quantum many-body effects. For example, significantly different quantum effects between the 1D δ -function interacting Bose and Fermi gases are seen clearly from the Bethe Ansatz solutions [3, 5].

On the other hand, anyons [24, 25] may also exist in both two and one dimension, obeying fractional statistics. For a 2D electron gas in the fractional quantum Hall (FQH) regime, the quasi-particles are charged anyons [26]. A more general description of quantum statistics is provided by Haldane exclusion statistics [27], which is a formulation of fractional statistics based on a generalized Pauli exclusion principle, now called generalized exclusion statistics [28, 29]. In 1D, a wavefunction with anyonic symmetry may also occur [30]. Anyons in 1D

acquire a multi-step function-like phase when two identical particles exchange their positions in the scattering process. This topological phase results in rich quantum effects in the 1D interacting model of anyons [31, 32, 33, 34, 35].

In this paper, celebrating the 75th anniversary of the Bethe Ansatz, we give a detailed and pedagogic derivation of the Bethe Ansatz solution for 1D δ -function interacting anyons, discussed for the first time by Kundu [30]. The ground state properties derived from the thermodynamic Bethe Ansatz are also presented.

This paper is set out as follows. In section II, we introduce the 1D interacting anyon model and show the reducibility of the field-theoretic Hamiltonian to a quantum mechanics many-body problem. The Bethe Ansatz solution is derived in section III. In section IV, we study the thermodynamic Bethe Ansatz equations (TBA). Section V is devoted to concluding remarks.

II. THE MODEL

We consider N anyons with δ -function interaction in 1D with hamiltonian [30]

$$H = \frac{\hbar^2}{2m} \int_0^L dx \partial \Psi^\dagger(x) \partial \Psi(x) + \frac{1}{2} g_{1D} \int_0^L dx \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \quad (1)$$

under periodic boundary conditions, with x a coordinate in length L . Here m denotes the atomic mass. Hereafter we set $\hbar = 2m = 1$ for convenience. The coupling constant g_{1D} is determined by $g_{1D} = \hbar^2 c / m$ where, at least for the Bose gas, the coupling strength c is tuned through an effective 1D scattering length a_{1D} via confinement in experiments. We also use a dimensionless coupling constant $\gamma = c/n$ to characterize different physical regimes of the anyon gas, where $n = N/L$ is the linear density. The operators $\Psi^\dagger(x)$ and $\Psi(x)$ are the creation and annihilation operators (or operator valued particle density) at point x satisfying the anyonic (equal-time) commutation relations

$$\begin{aligned} \Psi(x_1) \Psi^\dagger(x_2) &= e^{-i\kappa w(x_1, x_2)} \Psi^\dagger(x_2) \Psi(x_1) + \delta(x_1 - x_2) \\ \Psi(x_1) \Psi(x_2) &= e^{i\kappa w(x_1, x_2)} \Psi(x_2) \Psi(x_1), \\ \Psi^\dagger(x_1) \Psi^\dagger(x_2) &= e^{i\kappa w(x_1, x_2)} \Psi^\dagger(x_2) \Psi^\dagger(x_1). \end{aligned} \quad (2)$$

Here the multi-step function $w(x_1, x_2) = -w(x_2, x_1) = 1$ for order $x_1 > x_2$, with $w(x, x) = 0$. The anyonic phase $w(x, x) = 0$ for two colliding particles has bosonic signature at the point

$x_1 = x_2$. We note that in imposing the restriction $e^{i\kappa w(x,x)} = e^{i\pi}$ on the anyonic phase the above commutation relations give hard-core relations which have been constructed based on lattice integrable models [36] and also suggested in the anyon-Fermi mapping for the anyonic Tonks-Girardeau gas [34]. We shall see that this phase is not suitable for the continuous model of anyons. In contrast to the 1D interacting Bose gas [3], hamiltonian (1) exhibits both anyonic statistical and dynamical interactions, which can map to a 1D interacting Bose gas with multi- δ -function and momentum-dependent interactions [30]. The more general anyonic statistical and dynamical interactions result in a richer range of quantum effects than those of the 1D interacting Bose gas.

We first consider the corresponding equation of motion $-i\partial_t\Psi(x,t) = [H, \Psi(x,t)]$ via the time-dependent quantum fields, i.e., the nonlinear Schrödinger equation

$$i\partial_t\Psi(x,t) = -\partial_x^2\Psi(x,t) + g_{1D}\Psi^+(x,t)\Psi^2(x,t). \quad (3)$$

In order to appreciate the subtlety of the anyonic phase, the nonlinear Schrödinger equation can be calculated as

$$\begin{aligned} & \left[\int_0^L dx_1 \left(\partial_{x_1}\Psi^\dagger(x_1)\partial_{x_1}\Psi(x_1) + \frac{1}{2}g_{1D}(\Psi^\dagger(x_1))^2(\Psi(x_1))^2 \right), \Psi(x_2) \right] \\ = & \left[\int_0^L dx_1 \left(-\Psi^\dagger(x_1)\partial_{x_1}^2\Psi(x_1) + \frac{1}{2}g_{1D}\Psi^{\dagger 2}(x_1)\Psi^2(x_1) \right), \Psi(x_2) \right] \\ = & -\int_0^L dx_1 \left(e^{i\kappa w(x_1,x_2)}\Psi^\dagger(x_1)\Psi(x_2) - e^{-i\kappa w(x_2,x_1)}\Psi^\dagger(x_1)\Psi(x_2) - \delta(x_2 - x_1) \right) \partial_{x_1}^2\Psi(x_1) \\ & + \frac{1}{2}g_{1D} \int_0^L dx_1 \left[\Psi^{\dagger 2}(x_1)\Psi^2(x_1)\Psi(x_2) - \left(e^{-2i\kappa w(x_2,x_1)}\Psi^{\dagger 2}(x_1)\Psi(x_2) \right. \right. \\ & \left. \left. + e^{-i\kappa w(x_2,x_1)}\delta(x_2 - x_1)\Psi^\dagger(x_1) + e^{-2i\kappa w(x_2,x_1)}\delta(x_2 - x_1)\Psi^\dagger(x_1) \right) \Psi^2(x_1) \right] \\ = & \partial_{x_2}^2\Psi(x_2) - g_{1D}\Psi^\dagger(x_2)\Psi^2(x_2). \end{aligned} \quad (4)$$

In the above equation the time t has been omitted for simplicity. The properties $w(x,x) = 0$ and $w(x_1, x_2) = -w(x_2, x_1)$ have been used. We stress that the phases induced from exchanging operator positions in the terms $\Psi(x_1)\Psi(x_2)$ and $\Psi(x_2)\Psi^\dagger(x_1)$ are some fixed constants, depending on the relation between x_1 and x_2 . The derivative operator ∂_{x_1} then only acts on the field operators. Furthermore, if we choose $e^{i\kappa w(x,x)} = e^{i\pi}$, the last term in the above equation does not exist due to the cancellation of the terms $e^{-i\kappa w(x_2,x_1)}\delta(x_2 - x_1)\Psi^\dagger(x_1)$ and $e^{-2i\kappa w(x_2,x_1)}\delta(x_2 - x_1)\Psi^\dagger(x_1)$. This means that there is no δ -function interaction in the quantum mechanical many-body hamiltonian without internal spin degrees of freedom due

to the hard-core behaviour. For this reason we take the choice of phase $w(x, x) = 0$ in order to ensure the commutation relations (2). Under this requirement we can obtain the form of the nonlinear Schrödinger equation (3).

Following the way suggested for the 1D Bose gas [11], we define a Fock vacuum state $\Psi(x)|0\rangle = 0$. Thus the number operator \mathbf{N} and momentum operator \mathbf{P} are

$$\mathbf{N} = \int_0^L dx \Psi^\dagger(x) \Psi(x), \quad (5)$$

$$\mathbf{P} = i \int_0^L dx [\partial_x \Psi^\dagger(x)] \Psi(x). \quad (6)$$

To properly denote the anyonic phases in the eigenfunctions $|\Phi\rangle$ of hamiltonian (1), we assign particle coordinates x_i in the order $x_1 < x_2 < \dots < x_N$. Based on this assigned order the multi-step function $w(x_i, x_j)$ can be used to count phase alternation when two particles are interchanged, i.e., $w(x_i, x_j) = -w(x_j, x_i)$. The N -particle eigenstate is written as

$$|\Phi\rangle = \int_0^L dx^N e^{-i\frac{\kappa N}{2}} \chi(x_1 \dots x_N) \Psi^\dagger(x_1) \dots \Psi^\dagger(x_N) |0\rangle \quad (7)$$

where the Bethe Ansatz wavefunction is of the form

$$\chi(x_1 \dots x_N) = e^{-\frac{i\kappa}{2} \sum_{x_i < x_j}^N w(x_i, x_j)} \sum_P A(k_{P1} \dots k_{PN}) e^{i(k_{P1}x_1 + \dots + k_{PN}x_N)}. \quad (8)$$

Here the sum extends over all $N!$ permutations P . The order in which the particles are created in the N -particle eigenstate (7) incurs the phase factor in the wave function (8). Integration involves changes in the order of creating particles due to the permutation of coordinates. We can denote the permutation P as

$$P = \begin{pmatrix} 1 & 2 & \dots & N \\ P1 & P2 & \dots & PN \end{pmatrix}$$

which transforms the m -th object into the Pm -th position for $1 \leq m \leq N$ [14]. In general, for two different permutations we have

$$Q = \begin{pmatrix} 1 & 2 & \dots & N \\ Q1 & Q2 & \dots & QN \end{pmatrix} = \begin{pmatrix} P1 & P2 & \dots & PN \\ Q_{P1} & Q_{P2} & \dots & Q_{PN} \end{pmatrix}.$$

Therefore

$$QP = \begin{pmatrix} 1 & 2 & \dots & N \\ Q_{P1} & Q_{P2} & \dots & Q_{PN} \end{pmatrix}, \quad QP^{-1} = \begin{pmatrix} P1 & P2 & \dots & PN \\ Q_1 & Q_2 & \dots & Q_N \end{pmatrix}.$$

In order to understand the above expressions, we give the explicit form of the eigenstate (7) for $N = 3$ in Appendix A. The coefficients $A(k_{P1} \dots k_{PN})$ are obtained explicitly via the Bethe Ansatz approach in the next section. By comparing eqns (A2) and (A4) we see clearly that the phase factors in the multi-valued wavefunction (8) are diminished by those from permutations of the particles in the eigenstate $|\Phi\rangle$ such that the integrand in (7) is single valued and fully symmetric. We extracted a global phase factor $e^{-i\kappa N/2}$ in order to symmetrize the anyonic phase factor in the wave function (8) so that it has $\kappa \rightarrow \kappa + 4\pi$ symmetry. The eigenstate still has $\kappa \rightarrow \kappa + 2\pi$ symmetry. Moreover, it is easily seen that the wave function satisfies the anyonic symmetry [30]

$$\chi(\dots x_i \dots x_j \dots) = e^{-i\kappa(\sum_{k=i+1}^j w(x_i, x_k) - \sum_{k=i+1}^{j-1} w(x_j, x_k))} \chi(\dots x_j \dots x_i \dots). \quad (9)$$

Acting on the eigenstate (7) with the operator \mathbf{P} defined in (6) gives

$$\begin{aligned} \mathbf{P} |\Phi\rangle &= i \int_0^L dy \int_0^L dx^N e^{-i\frac{\kappa N}{2}} \chi(x_1 \dots x_N) [\partial_y \Psi^+(y)] \\ &\quad \times \sum_{k=1}^N e^{-i\kappa \sum_{i=1}^{k-1} w(y-x_i)} \Psi^\dagger(x_1) \dots \Psi^\dagger(x_{k-1}) \delta(y-x_k) \Psi^\dagger(x_{k+1}) \dots \Psi^\dagger(x_N) |0\rangle \\ &= i \int_0^L dx^N e^{-i\frac{\kappa N}{2}} \chi(x_1 \dots x_N) \\ &\quad \times \sum_{k=1}^N \Psi^\dagger(x_1) \dots \Psi^\dagger(x_{k-1}) \partial_{x_k} \Psi^\dagger(x_k) \Psi^\dagger(x_{k+1}) \dots \Psi^\dagger(x_N) |0\rangle \\ &= \int_0^L dx^N e^{-i\frac{\kappa N}{2}} \left\{ -i \sum_{k=1}^N \partial_{x_k} \chi(x_1 \dots x_N) \right\} \Psi^\dagger(x_1) \dots \Psi^\dagger(x_N) |0\rangle \end{aligned} \quad (10)$$

It therefore follows that we can obtain the quantum mechanical momentum operator as

$$\mathbf{P} = -i \sum_{k=1}^N \partial_{x_k}. \quad (11)$$

Similarly, by acting on the eigenstate (7) with the hamiltonian (1) the eigenvalue problem for hamiltonian (1), namely

$$H |\Phi\rangle = \int_0^L dx^N e^{-i\frac{\kappa N}{2}} H_N \chi(x_1 \dots x_N) \Psi^\dagger(x_1) \dots \Psi^\dagger(x_N) |0\rangle, \quad (12)$$

can be reduced to solving the quantum mechanical problem

$$H_N \chi(x_1 \dots x_N) = E \chi(x_1 \dots x_N), \quad (13)$$

where

$$H_N = -\frac{\hbar^2}{2m} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g_{1D} \sum_{1 \leq i < j \leq N} \delta(x_i - x_j). \quad (14)$$

The quantum mechanical hamiltonian (14) describes the 1D δ -function interacting quantum gas of N anyons confined in a periodic length L . The details of the above calculation are lengthy but straightforward. Without loss of generality, we give the details of the proof for the case $N = 3$ in Appendix B. Again, the subtlety of the anyonic phases is such that they just match the applicability of the Bethe Ansatz – from the eigenvalue problem of the field theoretic hamiltonian (1) to the quantum many-body problem (14). The next task is to determine the wavefunction by means of the Bethe Ansatz.

III. BETHE ANSATZ SOLUTION

In the preceding section, we showed the equivalence between the quantum field theoretic and quantum many-body problems. The Bethe Ansatz wavefunction (8) was written for the fundamental domain $0 \leq x_1 < x_2 < \dots < x_N \leq L$ [30, 32]. The wavefunction in other domains is an extension of the wavefunction (8) via the anyonic symmetry (9). The δ -function potential causes a jump in the derivative of the wavefunction in the eigenvalue equation (13). Changing the coordinates to centre of mass coordinates $X = (x_j + x_k)/2$ and $Y = x_j - x_k$ leads to the eigenvalue equation (13) in the form

$$\left\{ \left(-\frac{\partial^2}{\partial x_1^2} \dots - \frac{1}{2} \frac{\partial^2}{\partial X^2} - 2 \frac{\partial^2}{\partial Y^2} \dots - \frac{\partial^2}{\partial x_N^2} \right) + 2c\delta(Y) - E \right\} \chi(\dots x_i \dots x_j \dots) = 0. \quad (15)$$

Integrating both sides of this equation with respect to Y from $-\epsilon$ to $+\epsilon$ and taking the limit $\epsilon \rightarrow 0$ gives the discontinuity condition

$$\begin{aligned} & (\partial_{x_j} - \partial_{x_i}) \chi(x_1, \dots, x_i, x_j, \dots, x_N) \big|_{x_j=x_i+\epsilon} - (\partial_{x_j} - \partial_{x_i}) \chi(x_1, \dots, x_j, x_i, \dots, x_N) \big|_{x_j=x_i-\epsilon} \\ & = 2c\chi(x_1, \dots, x_i, x_j, \dots, x_N) \big|_{x_i=x_j}. \end{aligned} \quad (16)$$

on the derivative of the wavefunction. The two particle scattering relation can then be worked out directly from the wavefunction $\chi(x_1, \dots, x_N)$. This model does not have internal degrees of freedom. Therefore, it is not necessary to introduce extra coordinate indices to distinguish the coefficients in the wave function for different domains. We may write the wavefunction as

$$\chi(x_1 \dots x_N) = e^{-\frac{i\pi}{2}(w(x_i, x_j) + \sum_{x_l < x_m}^N w(x_l, x_m))} [\dots A(\dots k_i, k_j \dots) e^{i(\dots + k_i x_i + k_j x_j + \dots)}$$

$$+A(\dots k_j, k_i \dots) e^{i(\dots + k_j x_i + k_i x_j + \dots)} + \dots] \quad (17)$$

for the domain $0 \leq x_1 < x_2 < \dots x_i < x_j \dots < x_N \leq L$ and

$$\begin{aligned} \chi(x_1 \dots x_N) = & e^{-\frac{i\kappa}{2}(w(x_j, x_i) + \sum_{x_l < x_m}^N w(x_l, x_m))} [\dots A(\dots k_i, k_j \dots) e^{i(\dots + k_i x_j + k_j x_i + \dots)} \\ & \dots + A(\dots k_j, k_i \dots) e^{i(\dots + k_j x_j + k_i x_i + \dots)} + \dots] \end{aligned} \quad (18)$$

for the domain $0 \leq x_1 < x_2 < \dots x_j < x_i \dots < x_N \leq L$. For $x_i = x_j$, the wavefunction is given by

$$\begin{aligned} \chi(x_1 \dots x_N) = & e^{-\frac{i\kappa}{2}(w(x_i, x_i) + \sum_{x_l < x_m}^N w(x_l, x_m))} [\dots A(\dots k_i, k_j \dots) e^{i(\dots + (k_i + k_j)x_i + \dots)} \\ & + A(\dots k_j, k_i \dots) e^{i(\dots + (k_i + k_j)x_i + \dots)} + \dots]. \end{aligned} \quad (19)$$

Application of the derivative discontinuity condition (16) on the wavefunctions (17), (18) and (19) gives

$$\begin{aligned} & \left\{ e^{-\frac{1}{2}i\kappa w(x_i, x_j)} + e^{-\frac{1}{2}i\kappa w(x_j, x_i)} \right\} i(k_j - k_i) [A(\dots k_i, k_j \dots) - A(\dots k_j, k_i \dots)] \\ & = 2c [A(\dots k_i, k_j \dots) + A(\dots k_j, k_i \dots)]. \end{aligned} \quad (20)$$

This equation gives a relation between the coefficients $A(k_{P1} \dots k_{PN})$ of the form [30, 32]

$$A(\dots k_j, k_i \dots) = \frac{k_j - k_i + ic'}{k_j - k_i - ic'} A(\dots k_i, k_j \dots) \quad (21)$$

which is the two-body scattering relation. Here the anyonic parameter κ and the dynamical interaction c are inextricably related via the effective coupling constant [30, 32]

$$c' = c / \cos(\kappa/2). \quad (22)$$

This results in an interesting resonance-like effect in the effective coupling constant c' with respect to the statistical interaction around $\kappa = \pi$ [32]. The single-valued eigenstate (7) makes the continuity of the wavefunction (8) well defined because the phase factors in the multi-valued wavefunction (8) can be cancelled by those from permutations of the particles in the eigenstate $|\Phi\rangle$ for different domains. The probability density can be written as $|\chi(x_1, \dots, x_N)|^2$ which is subject to the scattering problem. It suffices to request a continuity relation of the wavefunction of the form

$$\lim_{x_i=x_j} \chi(x_1, \dots, x_i, x_j, \dots, x_N) = \lim_{x_i=x_j} \chi(x_1, \dots, x_j, x_i, \dots, x_N) = \chi(x_1, \dots, x_i, x_i, \dots, x_N), \quad (23)$$

or equivalently

$$\begin{aligned} & \lim_{x_i=x_j} [A(\dots k_i, k_j \dots) e^{i(\dots + k_i x_i + k_j x_j + \dots)} + A(\dots k_j, k_i \dots) e^{i(\dots + k_j x_i + k_i x_j + \dots)} + \dots] \\ &= \lim_{x_i=x_j} [A(\dots k_i, k_j \dots) e^{i(\dots + k_i x_j + k_j x_i + \dots)} + A(\dots k_j, k_i \dots) e^{i(\dots + k_j x_j + k_i x_i + \dots)} + \dots] \end{aligned} \quad (24)$$

which leads to an identity equation for the coefficients in the above two domains.

The periodic boundary conditions $\chi(x_1 = 0, x_2 \dots x_N) = \chi(x_2 \dots x_N, x_1 = L)$ for the system are equivalent to $\chi(x_1, x_2 \dots x_N) = e^{-i\kappa(N-1)} \chi(x_1 + L, x_2 \dots x_N)$. The anyonic phases for x_1 at the left end and at the right end of the domain are different due to the changes in the order of particles. Explicitly,

$$\begin{aligned} & e^{-\frac{i\kappa}{2}(\sum_{i=2}^N w(x_1, x_i) + \sum_{1 < i < j} w(x_i, x_j))} \sum_P A(k_{P1} \dots k_{PN}) e^{i(k_{P2}x_2 + \dots + k_{PN}x_N)} \\ &= e^{-\frac{i\kappa}{2}(\sum_{i=2}^N w(x_i, x_1) + \sum_{1 < i < j} w(x_i, x_j))} \sum_Q A(k_{Q1} \dots k_{QN}) e^{i(k_{Q1}x_2 + k_{Q2}x_3 + \dots + k_{Q(N-1)}x_N + k_{QN}L)} \end{aligned} \quad (25)$$

Notice that the second sum in the phase factors in both sides will cancel in deriving the Bethe ansatz equations below. In order to compare the coefficients on both sides of the above equation, we permute the order Q back to the order P . There are $N!$ patterns related to all different ways of displaying the positions of k 's. For example, in the right hand side of equation (25) we can permute k_1 back from the right side of

$$\begin{pmatrix} k_1 & k_2 & \dots & k_{N-1} & k_N \\ k_2 & k_3 & \dots & k_N & k_1 \end{pmatrix}$$

to the left side

$$\begin{pmatrix} k_1 & k_2 & \dots & k_{N-1} & k_N \\ k_1 & k_2 & \dots & k_{N-1} & k_N \end{pmatrix}.$$

The Bethe Ansatz equations (BAE)

$$e^{ik_j L} = -e^{i\kappa(N-1)} \prod_{\ell=1}^N \frac{k_j - k_\ell + i c'}{k_j - k_\ell - i c'} \quad (26)$$

follow directly through the scattering process. We demonstrate the derivation of the BAE (26) in Appendix C for $N = 3$.

It should be noted that this model reduces to the interacting Bose gas [3] for $\kappa = 0$. For $\kappa = \pi$ and 3π it reduces to the free Fermi gas. The anyonic gas lies in the range $0 \leq \kappa \leq \pi$ and $3\pi \leq \kappa \leq 4\pi$, where the effective interaction $c' > 0$. However, if the anyon parameter κ is tuned smoothly from $\kappa < \pi$ to $\kappa > \pi$, i.e., $\pi \leq \kappa \leq 3\pi$, the effective interaction is attractive. The ground state properties of this model have been studied in Ref. [32].

IV. THERMODYNAMIC BETHE ANSATZ

In general the exponential phase factor in the BAE (26), picked up from the statistical interaction during the scattering process, may shift the system into higher excited states. The total momentum is $p = N(N-1)\kappa/L + 2d\pi/L$, where d is an arbitrary integer. This can be easily seen from the logarithmic form

$$k_j L = 2\pi I_j + \kappa(N-1) - 2 \sum_{l=1}^N \arctan\left(\frac{k_j - k_l}{c'}\right) \quad (27)$$

of the BAE, where $I_j = j - (N+1)/2$ with $j = 1, \dots, N$. The energy is given by $E = \sum_{j=1}^N k_j^2$.

In minimizing the energy we consider $\kappa(N-1) = \nu \pmod{2\pi}$ in the phase factor with $-\pi \leq \nu \leq \pi$. Each quasimomentum k_j shifts to $k_j + \nu/L$ in the ground state. In general the anyonic factor κ produces an effect like the self-sustained Aharonov-Bohm flux resulting in metastable states. In the thermodynamic limit, the BAE (26) can be written in the form of an integral equation

$$\rho(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-Q}^Q \frac{2c'}{c'^2 + (k - k')^2} \rho(k') dk'. \quad (28)$$

Here Q is the cut-off momentum for the ground state. We have introduced the density of roots $\rho(k_j) = 1/(L(k_{j+1} - k_j))$ in the interval $k_{j+1} - k_j$. It follows that in the thermodynamic limit the lowest energy is given by $E = N(n^2 e(\gamma, \kappa) + \nu^2/L^2)$ where

$$e(\gamma, \kappa) = \frac{\gamma^3}{\lambda^3} \int_{-1}^1 g(x) x^2 dx. \quad (29)$$

The last term in the energy E can be ignored compared to the kinetic energy. The root density $g(x) := \rho(Qx)$ and the parameter $\lambda = c/Q$ are determined by Lieb-Liniger type integral equations of the form

$$\begin{aligned} g(x) &= \frac{1}{2\pi} + \frac{\lambda \cos(\kappa/2)}{\pi} \int_{-1}^1 \frac{g(y) dy}{\lambda^2 + \cos^2(\frac{\kappa}{2})(x - y)^2} \\ \lambda &= \gamma \int_{-1}^1 g(x) dx. \end{aligned} \quad (30)$$

The numerical solution of this equation has been given in Ref. [32].

At zero temperature, the quantum numbers I_j form a uniform lattice from $-(N-1)/2$ to $(N-1)/2$. The quasimomenta k_j form a non-uniform distribution $\rho(k)$ between $k = -Q$ and $k = Q$, which is determined by the BAE (28). The occupied lattice sites are called

quasiparticles. For the excited states the lattice sites are not all occupied. This means that some k 's move out of the Dirac sea leaving some lattice sites unoccupied, the so called holes. If one k leaves the Dirac sea, the remaining roots all move. This collective behaviour phenomenon is governed by the linear dispersion relation $E - E_0 \approx v_c p$ as $p \rightarrow 0$, where the sound velocity is $v_c \approx 2n\pi(1 - 4\gamma^{-1} \cos(\kappa/2))$ [32]. In this way, the density of quasiparticles and the density of the holes are determined by the equation

$$\rho(k) + \rho_h(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2c' \rho(k') dk'}{c'^2 + (k - k')^2}. \quad (31)$$

The energy per particle for the state is given by

$$\frac{E}{N} = n^{-1} \int_{-\infty}^{\infty} \rho(k) k^2 dk \quad (32)$$

where the linear density $n = \int_{-\infty}^{\infty} \rho(k) dk$.

To obtain the thermodynamics following the Yang-Yang approach [37], an equilibrium state associated with the entropy

$$\frac{S}{N} = n^{-1} \int_{-\infty}^{\infty} [(\rho(k) + \rho_h(k)) \ln(\rho(k) + \rho_h(k)) - \rho(k) \ln \rho(k) - \rho_h(k) \ln \rho_h(k)] dk \quad (33)$$

is a mixture of some eigenstates of the Hamiltonian. For the equilibrium state the density of quasiparticles is determined by minimizing the free energy $F = E - TS - \mu N$, i.e. $\delta F = 0$, where μ is the chemical potential. It follows that the TBA equation and the free energy are given by

$$\epsilon(k) = \epsilon^0(k) - \mu - \frac{T}{2\pi} \int_{-\infty}^{\infty} dk' \theta'(k - k') \ln(1 + e^{-\frac{\epsilon(k')}{T}}) \quad (34)$$

$$\frac{F}{L} = \mu n - \frac{T}{2\pi} \int_{-\infty}^{\infty} dk \ln(1 + e^{-\frac{\epsilon(k)}{T}}) \quad (35)$$

respectively. Here

$$\theta'(x) = \frac{2c \cos(\kappa/2)}{c^2 + \cos^2(\kappa/2) x^2} \quad (36)$$

Here $\epsilon(k)$ is the dressed energy $e^{\epsilon(k)/T} := \rho^h(k)/\rho(k)$ and the function $\epsilon^0(k) = (k + \nu/L)^2$. The thermodynamics of the system has been discussed in Ref. [33]. For $\kappa = 0$, the thermodynamics of the 1D Bose gas has been studied extensively, see, e.g. Refs [38, 39].

The TBA equation (34) also provides ground state properties and a clear picture of the band filling. At zero temperature, the negative part ϵ^- of the dressed energy makes a contribution to the free energy, i.e. for $T \rightarrow 0$, the TBA equation (34) becomes

$$\epsilon(k) = k^2 - \mu + \frac{1}{2\pi} \int_{-Q}^Q \frac{2c' \epsilon^-(k')}{c'^2 + (k - k')^2} dk'. \quad (37)$$

The pressure per unit length is given by $P_0 = -\frac{1}{2\pi} \int_{-Q}^Q \epsilon^-(k) dk$. For the Tonks-Girardeau regime, i.e. for $\gamma \gg 1$, we have

$$\epsilon(k) \approx k^2 - \mu - \frac{2P_0 c'}{c'^2 + k^2}. \quad (38)$$

From this equation, with the help of the conditions $\epsilon(Q) = 0$ and $n = \partial P_0 / \partial \mu$, one can obtain physical quantities, such as the ground state energy E_0 , chemical potential μ , pressure P_0 and the cut-off momentum Q . The results are

$$\begin{aligned} E_0/L &\approx \frac{1}{3} n^3 \pi^2 \left(1 - \frac{4 \cos(\kappa/2)}{\gamma} \right), \quad \mu \approx n^2 \pi^2 \left(1 - \frac{16 \cos(\kappa/2)}{3\gamma} \right), \\ P_0 &\approx \frac{2}{3} n^3 \pi^2 \left(1 - \frac{6 \cos(\kappa/2)}{\gamma} \right), \quad Q \approx n\pi \left(1 - \frac{2 \cos(\kappa/2)}{\gamma} \right). \end{aligned} \quad (39)$$

The macroscopic velocity is

$$v = \sqrt{2 \frac{\partial P_0}{\partial n}} \approx 2n\pi(1 - 4\gamma^{-1} \cos(\kappa/2)). \quad (40)$$

On the other hand, from the BAE (28), we have

$$\rho(k) \approx \frac{1}{2\pi} \left(1 + \frac{2nc'}{c'^2 + k^2} + \frac{4c'kp}{(c'^2 + k^2)^2} \right), \quad (41)$$

where p is the total momentum. Then from the condition $n = \int_{-Q}^Q \rho(k) dk$, we have

$$Q \approx n\pi \left(1 - \frac{2 \cos(\kappa/2)}{\gamma} \right), \quad E_0/L \approx \frac{1}{3} n^3 \pi^2 \left(1 - \frac{4 \cos(\kappa/2)}{\gamma} + \frac{12 \cos^2(\kappa/2)}{\gamma^2} \right) \quad (42)$$

which coincide with the TBA results (39).

V. CONCLUSION

In this paper we have derived the Bethe Ansatz solution of the 1D δ -function interacting model of anyons, which is a natural extension of the 1D δ -function interacting Bose gas, recovered in the limit $\kappa = 0$. The ground state properties (39) have been derived from the TBA equations (34). The subtlety of the anyonic symmetry results in rich quantum effects for the degenerate anyon gas [32, 33]. Here we remark that a generalization of this model to the 1D δ -function interacting model of anyons with internal spin degrees of freedom should be possible. We expect that the combination of anyonic statistical interaction, dynamical interaction and spin degrees will result in new quantum effects.

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APPENDIX A: THREE PARTICLE EIGENSTATE

Here we give the explicit eigenstate (7) for $N = 3$. For the order of particle coordinates $x_1 < x_2 < x_3$ the wavefunction is

$$\chi(x_1, x_2, x_3) = e^{-i\frac{\kappa}{2}(w(x_1, x_2) + w(x_1, x_3) + w(x_2, x_3))} \sum_P A(k_{P1}, k_{P2}, k_{P3}) e^{k_{P1}x_1 + k_{P2}x_2 + k_{P3}x_3}. \quad (\text{A1})$$

The sum runs over all $3!$ permutations P . According to the definition in the anyonic commutation relations the above phase factor is $e^{i\frac{3}{2}\kappa}$ for $N = 3$ in the assigned order $x_1 < x_2 < x_3$. In this domain the eigenstate for hamiltonian (1) is

$$\begin{aligned} |\Phi\rangle &= \int dx^3 e^{-i\frac{3\kappa}{2}} e^{-i\frac{\kappa}{2}(w(x_1, x_2) + w(x_1, x_3) + w(x_2, x_3))} \left[\sum_P A(k_{p1}, k_{p2}, k_{p3}) e^{k_{P1}x_1 + k_{P2}x_2 + k_{P3}x_3} \right] \\ &\quad \times \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\ &= \int dx^3 \left[\sum_P A(k_{p1}, k_{p2}, k_{p3}) e^{k_{P1}x_1 + k_{P2}x_2 + k_{P3}x_3} \right] \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle. \end{aligned} \quad (\text{A2})$$

The phase factor is unity in the assigned order $x_1 < x_2 < x_3$.

Now consider two-particle exchange, i.e. the particle order $x_2 < x_1 < x_3$. We can write the wavefunction as

$$\chi(x_2, x_1, x_3) = e^{-i\frac{\kappa}{2}(w(x_2, x_1) + w(x_2, x_3) + w(x_1, x_3))} \left[\sum_P A(k_{P1}, k_{P2}, k_{P3}) e^{k_{P1}x_2 + k_{P2}x_1 + k_{P3}x_3} \right] \quad (\text{A3})$$

Thus the eigenstate is given by

$$\begin{aligned} |\Phi\rangle &= \int dx^3 e^{-i\frac{3\kappa}{2}} e^{-i\frac{\kappa}{2}(w(x_2, x_1) + w(x_2, x_3) + w(x_1, x_3))} \left[\sum_P A(k_{P1}, k_{P2}, k_{P3}) e^{k_{P1}x_2 + k_{P2}x_1 + k_{P3}x_3} \right] \\ &\quad \times \Psi^\dagger(x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_3) |0\rangle \\ &= \int dx^3 e^{-i\frac{3\kappa}{2}} e^{-i\frac{\kappa}{2}(w(x_1, x_2) + w(x_2, x_3) + w(x_1, x_3))} \left[\sum_P A(k_{P1}, k_{P2}, k_{P3}) e^{k_{P1}x_2 + k_{P2}x_1 + k_{P3}x_3} \right] \\ &\quad \times \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\ &= \int dx^3 \left[\sum_P A(k_{P1}, k_{P2}, k_{P3}) e^{k_{P1}x_2 + k_{P2}x_1 + k_{P3}x_3} \right] \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle. \end{aligned} \quad (\text{A4})$$

It is clear that the summation in the phase factor of the wavefunction χ simply counts the changes of the order in the relevant domain.

APPENDIX B: FROM QUANTUM FIELD TO MANY-BODY QUANTUM MECHANICS: $N = 3$

In this appendix we explicitly demonstrate the relation between the quantum field theoretic hamiltonian and the quantum mechanics many-body problem. First, we apply the periodic conditions and integrate by parts in the first part of hamiltonian (1)

$$\int_0^L dx [\partial_x \Psi^\dagger(x)] [\partial_x \Psi(x)] = - \int_0^L dx (\partial_x^2 \Psi^\dagger(x)) \Psi(x). \quad (\text{B1})$$

For $N = 3$, the N -particle eigenstate $|\Phi\rangle$ in (7) reads

$$|\Phi\rangle = \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle, \quad x_1 < x_2 < x_3. \quad (\text{B2})$$

We consider operations H on this state. Let $A = \int_0^L dx [\partial_x \Psi^\dagger(x)] [\partial_x \Psi(x)] |\Phi\rangle$, then

$$\begin{aligned} A &= - \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \Psi(x) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\ &= - \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) (e^{-i\kappa w(x, x_1)} \Psi^\dagger(x_1) \Psi(x) + \delta(x - x_1)) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\ &= - \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \delta(x - x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\ &\quad - \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) e^{-i\kappa w(x, x_1)} \Psi^\dagger(x_1) \Psi(x) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle. \end{aligned} \quad (\text{B3})$$

Here we succeeded in exchanging the position for $\Psi(x)$ and $\Psi^\dagger(x_1)$ by means of the commutation relations (2). In (B3), the first term is the right term. We continue this procedure for the second term until $\Psi(x)$ is moved to the right hand side of $\Psi^\dagger(x_3)$. Thus

$$\begin{aligned} A &= - \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \delta(x - x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\ &\quad - \int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1))} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \Psi^\dagger(x_1) (e^{-i\kappa w(x, x_2)} \Psi^\dagger(x_2) \Psi(x) + \delta(x - x_2)) \Psi^\dagger(x_3) |0\rangle \end{aligned}$$

$$\begin{aligned}
&= -\int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \delta(x - x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\
&\quad -\int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1))} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \Psi^\dagger(x_1) \delta(x - x_2) \Psi^\dagger(x_3) |0\rangle \\
&\quad -\int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1) + \kappa w(x, x_2))} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \Psi^\dagger(x_1) (\Psi^\dagger(x_2) \Psi(x)) \Psi^\dagger(x_3) |0\rangle \\
&= -\int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \delta(x - x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\
&\quad -\int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1))} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \Psi^\dagger(x_1) \delta(x - x_2) \Psi^\dagger(x_3) |0\rangle \\
&\quad -\int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1) + \kappa w(x, x_2))} \chi(x_1, x_2, x_3) (\partial_x^2 \Psi^\dagger(x)) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \delta(x - x_3) |0\rangle.
\end{aligned}$$

The results $\Psi(x) \Psi^\dagger(x_3) = e^{-i\kappa w(x, x_3)} \Psi^\dagger(x_3) \Psi(x) + \delta(x - x_3)$ and $\Psi(x) |0\rangle = 0$ are used in the last step. Using integration by parts and the integral property of the δ -function, we then have

$$\begin{aligned}
A &= -\int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} (\partial_{x_1}^2 \chi(x_1, x_2, x_3)) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\
&\quad -\int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_2, x_1))} (\partial_{x_2}^2 \chi(x_1, x_2, x_3)) \Psi^\dagger(x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_3) |0\rangle \\
&\quad -\int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_3, x_1) + \kappa w(x_3, x_2))} (\partial_{x_3}^2 \chi(x_1, x_2, x_3)) \Psi^\dagger(x_3) \Psi^\dagger(x_1) \Psi^\dagger(x_2) |0\rangle. \\
&= -\int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} ((\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2) \chi(x_1, x_2, x_3)) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle. \quad (\text{B4})
\end{aligned}$$

Here the commutation relations (2) are used to form the pattern $\Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3)$ by moving around $\Psi^\dagger(x_2)$ and $\Psi^\dagger(x_3)$.

We now turn to the second part of the hamiltonian (1). Accordingly, let $B =$

$\int_0^L dx \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) | \Phi \rangle$, then

$$\begin{aligned}
B &= \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) | 0 \rangle \\
&= \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\Psi^\dagger(x))^2 \Psi(x) (\Psi^\dagger(x_1) \Psi(x) e^{-i\kappa w(x, x_1)} + \delta(x - x_1)) \Psi^\dagger(x_2) \Psi^\dagger(x_3) | 0 \rangle \\
&= \int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x))^2 \Psi(x) \Psi^\dagger(x_1) \Psi(x) \Psi^\dagger(x_2) \Psi^\dagger(x_3) | 0 \rangle \\
&\quad + \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\Psi^\dagger(x))^2 \Psi(x) \delta(x - x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) | 0 \rangle \\
&= \int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x))^2 \Psi(x) \Psi^\dagger(x_1) (\Psi^\dagger(x_2) \Psi(x) e^{-i\kappa w(x, x_2)} + \delta(x - x_2)) \Psi^\dagger(x_3) | 0 \rangle \\
&\quad + \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_1))^2 \Psi(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) | 0 \rangle \\
&= \int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1)) + \kappa w(x, x_2)} \chi(x_1, x_2, x_3) (\Psi^\dagger(x))^2 \Psi(x) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi(x) \Psi^\dagger(x_3) | 0 \rangle \\
&\quad + \int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x))^2 \Psi(x) \Psi^\dagger(x_1) \delta(x - x_2) \Psi^\dagger(x_3) | 0 \rangle \\
&\quad + \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_1))^2 (\Psi^\dagger(x_2) \Psi(x_1) e^{-i\kappa w(x_1, x_2)} + \delta(x_1 - x_2)) \Psi^\dagger(x_3) | 0 \rangle \\
&= \int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x, x_1)) + \kappa w(x, x_2)} \chi(x_1, x_2, x_3) (\Psi^\dagger(x))^2 \Psi(x) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \delta(x - x_3) | 0 \rangle \\
&\quad + \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_2, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_2))^2 \Psi(x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_3) | 0 \rangle \\
&\quad + \int_0^L dx dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_1, x_2))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_1))^2 \Psi^\dagger(x_2) \Psi(x_1) \Psi^\dagger(x_3) | 0 \rangle \\
&\quad + \int_0^L dx dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_1))^2 \delta(x_1 - x_2) \Psi^\dagger(x_3) | 0 \rangle
\end{aligned}$$

$$\begin{aligned}
&= \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_3, x_1)) + \kappa w(x_3, x_2)} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_3))^2 \Psi(x_3) \Psi^\dagger(x_1) \Psi^\dagger(x_2) |0\rangle \\
&+ \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_2, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_2))^2 \Psi(x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_3) |0\rangle \\
&+ \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_1, x_2))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_1))^2 \Psi^\dagger(x_2) (\Psi^\dagger(x_3) \Psi(x_1) e^{-i\kappa w(x_1, x_3)} + \delta(x_1 - x_3)) |0\rangle \\
&+ \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_1))^2 \delta(x_1 - x_2) \Psi^\dagger(x_3) |0\rangle. \tag{B5}
\end{aligned}$$

Denote these four last terms by b_4 , b_3 , b_2 and b_1 in order from top to bottom.

We now discuss these terms case by case. The simplest case is b_1 . With the help of the property of the δ function it becomes

$$b_1 = \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) \delta(x_1 - x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \tag{B6}$$

as expected. By using $\Psi(x) |0\rangle = 0$, we have

$$\begin{aligned}
b_2 &= \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_1, x_2))} \chi(x_1, x_2, x_3) \delta(x_1 - x_3) (\Psi^\dagger(x_1))^2 \Psi^\dagger(x_2) |0\rangle \\
&= \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_1, x_2))} \chi(x_1, x_2, x_3) \delta(x_1 - x_3) \Psi^\dagger(x_1) \Psi^\dagger(x_3) \Psi^\dagger(x_2) |0\rangle \\
&= \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_1, x_2))} \chi(x_1, x_2, x_3) \delta(x_1 - x_3) \Psi^\dagger(x_1) e^{i\kappa w(x_3, x_2)} \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\
&= \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) \delta(x_1 - x_3) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle. \tag{B7}
\end{aligned}$$

The term b_3 is more complicated because we have to move $\Psi(x_2)$ to the right hand side of $\Psi^\dagger(x_3)$, namely

$$\begin{aligned}
b_3 &= \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2} + w(x_2, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_2))^2 (\Psi^\dagger(x_1) \Psi(x_2) e^{-i\kappa w(x_2, x_1)} + \delta(x_1 - x_2)) \Psi^\dagger(x_3) |0\rangle \\
&= \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + 2\kappa w(x_2, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_2))^2 \Psi^\dagger(x_1) \Psi(x_2) \Psi^\dagger(x_3) |0\rangle
\end{aligned}$$

$$\begin{aligned}
& + \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + \kappa w(x_2, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_2))^2 \delta(x_1 - x_2) \Psi^\dagger(x_3) |0\rangle \\
& = \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + 2\kappa w(x_2, x_1))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_2))^2 \Psi^\dagger(x_1) (\Psi^\dagger(x_3) \Psi(x_2) e^{-i\kappa w(x_2, x_3)} + \delta(x_2 - x_3)) |0\rangle \\
& + \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) \delta(x_1 - x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\
& = \int_0^L dx_1 dx_2 dx_3 e^{-i(\frac{3}{2}\kappa + 2\kappa w(x_2, x_1))} \chi(x_1, x_2, x_3) \Psi^\dagger(x_2) \Psi^\dagger(x_3) \Psi^\dagger(x_1) \delta(x_2 - x_3) |0\rangle \\
& + \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) \delta(x_1 - x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle.
\end{aligned}$$

Now using commutation relations (2), we move $\Psi^\dagger(x_1)$ to the left side of $\Psi^\dagger(x_2)$ such that

$$\begin{aligned}
b_3 & = \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) \delta(x_2 - x_3) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\
& + \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) \delta(x_1 - x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle.
\end{aligned} \tag{B8}$$

Now consider the term b_4 . Similar to the term b_3 , we have to move $\Psi(x_3)$ to the right hand side of $\Psi^\dagger(x_2)$ in order to form the state $\Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle$. We thus have

$$\begin{aligned}
b_4 & = \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2} + w(x_3, x_1) + w(x_3, x_2))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_3))^2 (\Psi^\dagger(x_1) \Psi(x_3) e^{-i\kappa w(x_3, x_1)} + \delta(x_3 - x_1)) \Psi^\dagger(x_2) |0\rangle \\
& = \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2} + 2w(x_3, x_1) + w(x_3, x_2))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_3))^2 \Psi^\dagger(x_1) \Psi(x_3) \Psi^\dagger(x_2) |0\rangle \\
& + \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2} + w(x_3, x_1) + w(x_3, x_2))} \chi(x_1, x_2, x_3) \delta(x_3 - x_1) (\Psi^\dagger(x_3))^2 \Psi^\dagger(x_2) |0\rangle \\
& = \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2} + 2w(x_3, x_1) + w(x_3, x_2))} \chi(x_1, x_2, x_3) (\Psi^\dagger(x_3))^2 \Psi^\dagger(x_1) (\Psi^\dagger(x_2) \Psi(x_3) e^{-i\kappa w(x_3, x_2)} + \delta(x_3 - x_2)) |0\rangle \\
& + \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2} + w(x_3, x_2))} \chi(x_1, x_2, x_3) \delta(x_3 - x_1) \Psi^\dagger(x_1) \Psi^\dagger(x_3) \Psi^\dagger(x_2) |0\rangle
\end{aligned}$$

$$\begin{aligned}
&= \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2}+2w(x_3,x_1)+w(x_3,x_2))} \chi(x_1,x_2,x_3) \delta(x_3-x_2) (\Psi^\dagger(x_3)) \Psi^\dagger(x_1) |0\rangle \\
&+ \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2}+w(x_3,x_2))} \chi(x_1,x_2,x_3) \delta(x_3-x_1) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) e^{i\kappa w(x_3,x_2)} |0\rangle \\
&= \int_0^L dx_1 dx_2 dx_3 e^{-i\kappa(\frac{3}{2}+2w(x_3,x_1))} \chi(x_1,x_2,x_3) \delta(x_3-x_2) \Psi^\dagger(x_2) \Psi^\dagger(x_3) \Psi^\dagger(x_1) |0\rangle \\
&+ \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1,x_2,x_3) \delta(x_3-x_1) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle
\end{aligned}$$

Therefore

$$\begin{aligned}
b_4 &= \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1,x_2,x_3) \delta(x_3-x_2) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\
&+ \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1,x_2,x_3) \delta(x_3-x_1) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle. \tag{B9}
\end{aligned}$$

Summing the individual results for b_1 (B6), b_2 (B7) b_3 (B8) and b_4 (B9) gives the result

$$B = 2 \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \chi(x_1, x_2, x_3) (\delta(x_1-x_2) + \delta(x_1-x_3) + \delta(x_2-x_3)) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle. \tag{B10}$$

Finally, we have

$$\begin{aligned}
H | \Phi \rangle &= \frac{\hbar^2}{2m} A + \frac{1}{2} g_{1D} B \\
&= \int_0^L dx_1 dx_2 dx_3 e^{-i\frac{3}{2}\kappa} \left(-\frac{\hbar^2}{2m} \sum_{i=1}^3 \frac{\partial^2 \chi}{\partial x_i^2} + g_{1D} \sum_{1 \leq i < j \leq 3} \delta(x_i - x_j) \chi \right) \\
&\quad \times \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi^\dagger(x_3) |0\rangle \\
&= E | \Phi \rangle \tag{B11}
\end{aligned}$$

provided χ satisfies

$$-\frac{\hbar^2}{2m} \sum_{i=1}^3 \frac{\partial^2 \chi}{\partial x_i^2} + g_{1D} \sum_{1 \leq i < j \leq 3} \delta(x_i - x_j) \chi = E \chi, \tag{B12}$$

which is the desired three-particle quantum mechanics problem. Here $\chi = \chi(x_1, x_2, x_3)$ and E is the eigenvalue.

APPENDIX C: PERIODIC BOUNDARY CONDITIONS

Application of the periodic boundary condition on the Bethe wavefunctions for $N = 3$ gives

$$\chi(x_1 = 0, x_2, x_3) = e^{-i\frac{\kappa}{2}(w(x_1, x_2) + w(x_1, x_3) + w(x_2, x_3))} \sum_P A(k_{P1}, k_{P2}, k_{P3}) e^{i(k_{P2}x_2 + k_{P3}x_3)} \quad (C1)$$

$$\chi(x_2, x_3, x_1 = L) = e^{-i\frac{\kappa}{2}(w(x_2, x_3) + w(x_2, x_1) + w(x_3, x_1))} \sum_Q A(k_{Q1}, k_{Q2}, k_{Q3}) e^{i(k_{Q1}x_2 + k_{Q2}x_3 + k_{Q3}L)} \quad (C2)$$

We thus have

$$\begin{aligned} e^{i\frac{\kappa}{2}(2w(x_1, x_2) + 2w(x_1, x_3))} \sum_P A(k_{P1}, k_{P2}, k_{P3}) e^{i(k_{P2}x_2 + k_{P3}x_3)} \\ = \sum_Q A(k_{Q1}, k_{Q2}, k_{Q3}) e^{i(k_{Q1}x_2 + k_{Q2}x_3 + k_{Q3}L)} \end{aligned} \quad (C3)$$

Namely

$$\begin{aligned} e^{2i\kappa} \sum_P A(k_{P1}, k_{P2}, k_{P3}) e^{k_{P2}x_2 + k_{P3}x_3} \\ = \sum_Q A(k_{Q1}, k_{Q2}, k_{Q3}) e^{k_{Q1}x_2 + k_{Q2}x_3 + k_{Q3}L} \end{aligned} \quad (C4)$$

according the order $x_1 < x_2 \dots < x_N$. The condition (C4) gives six equations. Only three of them

$$e^{ik_1L} = e^{i2\kappa} \frac{A(k_1 k_2 k_3)}{A(k_2 k_3 k_1)} = e^{i2\kappa} \frac{(k_1 - k_2 + ic')(k_1 - k_3 + ic')}{(k_1 - k_2 - ic')(k_1 - k_3 - ic')} \quad (C5)$$

$$e^{ik_2L} = e^{i2\kappa} \frac{A(k_2 k_1 k_3)}{A(k_1 k_3 k_2)} = e^{i2\kappa} \frac{(k_2 - k_1 + ic')(k_2 - k_3 + ic')}{(k_2 - k_1 - ic')(k_2 - k_3 - ic')} \quad (C6)$$

$$e^{ik_3L} = e^{i2\kappa} \frac{A(k_3 k_1 k_2)}{A(k_1 k_2 k_3)} = e^{i2\kappa} \frac{(k_3 - k_1 + ic')(k_3 - k_2 + ic')}{(k_3 - k_1 - ic')(k_3 - k_2 - ic')} \quad (C7)$$

are independent. Similarly consideration of the case $\chi(x_1, x_2, x_3) = e^{i2\kappa} \chi(x_1, x_2 + L, x_3)$ gives the same set of equations.